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THEORY OF THE MOTION OF SYSTEMS WITH ROLLING*

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A mathematical model is proposed for describing the motions of a system with rolling, and with or without slippage. Conditions are given for the transition from one mode of motion to another. Examples are included.

Rolling without slippage is equivalent to determining a kinematic constraint, generally rheonomic /1/, described by differential equations linear with respect to the generalized velocities. The equations cannot usually be reduced to finite relations connecting the generalized coordinates, and therefore rolling without slippage represents a motion with a non-holonomic constraint. Study of the motion of a system with rolling, taking slippage into account, reduces to the study of the dynamics of a system with releasing kinematic constraints. Two problems arise in this context.

1°. Using differential equations to describe the motion of a system with rolling in the general case;

2°. Establishing the conditions for transferring from one rolling mode to another.

In the classical mechanics of non-holonomic systems where rolling without slippage is usually discussed, the second problem disappears, and the first problem was solved by Chaplygin, Voronets, Boltzmann, Hamel, et al. When a wheel with an elastic deformable type rolls without slippage, kinematic constraints appear which differ considerably from the classical non-holonomic constraints arising when a rigid body is rolling. The general equations of motion of a wheeled carriage executing small deviations from its uniform rectilinear motion were given in /2/, where the Keldysh theorem concerning the rolling motion of a wheel with an elastic tyre was used. The equations were generalized in /3/ to the case of the curvilinear motion of a wheeled carriage along a trajectory of fairly small curvature.

In general, the equations of motion of a system with rolling have the simplest form in the moving coordinate system /4, 5/ and must be written in the form of equations in quasi-coordinates. As we know, the equations of motion of a non-holonomic system are also written in this form /6/, therefore the equations in quasicoordinates are the most suitable for describing the motion of a system with rolling, with or without slippage. We must however generalize the well-known Boltzmann-Hamel equations to the case of a system with rheonomic kinematic constraints. The equations in quasicoordinates obtained in this manner solve the first of the above problems and can be used as a basis for the general theory of the motion of systems with rolling.

Investigation of the structural features of the phase space of a system with rolling also enables the second problem to be solved. It also becomes clear that the equations of kinematic constraints describing rolling without slippage can be regarded as the equations of some hypersurface Π in phase space. For the case of rolling without slippage we have the corresponding motion of a phase point along the surface Π in the region stable with respect to deviations from the surface Π. By determining the boundaries of this region we can solve the problem of the conditions governing the passage from rolling without slippage to rolling with slippage, and we can find the conditions for the reverse process to occur.

1. General equations of dynamics for a system with rolling. Let the position of the system with rolling be defined by n generalized coordinates $q_1, q_2, \ldots, q_n$, and rolling without slippage by $n-m$ equations of the form

$$a_{ls}(q, t)\, q_s^{\cdot} + a_l(q, t) = 0 \qquad (1.1)$$
$$(l = m+1,\ m+2, \ldots, n)$$

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Here and henceforth the repeated indices denote summation from 1 to n.
Let us write the D'Alembert-Lagrange equation

$$\left(\frac{d}{dt}\frac{\partial T}{\partial q_s^{\cdot}} - \frac{\partial T}{\partial q_s} - Q_s\right)\delta q_s = 0 \tag{1.2}$$

($T = T(q, q^{\cdot}, t)$ is the kinetic energy of the system and $Q_s(q, q^{\cdot}, t)$ are given generalized forces). We introduce the quasicoordinates $\pi_1, \pi_2, \ldots, \pi_n$, whose time derivatives are connected with the generalized coordinates by the following independent relations:

$$\pi_j^{\cdot} = a_{js}(q, t)\, q_s^{\cdot} + a_j(q, t) \tag{1.3}$$

so that the right-hand sides of the last $n - m$ relations are identical with the left-hand sides of Eqs.(1.1). Solving the set of equations (1.3) for $q_1^{\cdot}, q_2^{\cdot}, \ldots, q_n^{\cdot}$, we obtain

$$q_s^{\cdot} = b_{si}(q, t)\, \pi_i^{\cdot} + b_s(q, t) \tag{1.4}$$

It can be confirmed that the coefficients in (1.3) and (1.4) are connected by the following relations:

$$a_{js}b_{si} = \delta_{ij},\ a_j = -a_{js}b_s,\ b_{si}a_{ir} = \delta_{sr} \tag{1.5}$$

in which δ_{ij} and δ_{sr} are the Kronecker deltas. According to (1.4) the variations of the true coordinates δq_s and quasicoordinates $\delta\pi_i$ are connected by the relations $\delta q_s = b_{si}(q, t)\,\delta\pi_i$. Substituting these into (1.2), we obtain a sum which, because of the independent nature of the variations $\delta\pi_i$, decomposes into n equations of the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_s^{\cdot}}\right) b_{si} - \frac{\partial T}{\partial q_s} b_{si} = Q_s b_{si} \tag{1.6}$$

The quantity $\Pi_i = Q_s b_{si}$ represents the generalized force which performs work on the displacement $\delta\pi_i$ provided that all the remaining $\delta\pi$ are zero. We shall introduce the function $T^* = T^*(q, \pi^{\cdot}, t)$ obtained from the expression for the kinetic energy $T(q, q^{\cdot}, t)$ after eliminating all $q_1^{\cdot}, q_2^{\cdot}, \ldots, q_n^{\cdot}$ with help of relations (1.4). Carrying out an inverse operation on the function $T^*(q, \pi^{\cdot}, t)$ obtained using relations (1.3) we clearly again obtain $T(q, q^{\cdot}, t)$, i.e.

$$T^*(q,\ a_{jr}q_r^{\cdot} + a_j,\ t) = T(q, q^{\cdot}, t) \tag{1.7}$$

Differentiating this relation with respect to $q_s^{\cdot}$, we have

$$\frac{\partial T}{\partial q_s^{\cdot}} = \frac{\partial T^*}{\partial \pi_j^{\cdot}} a_{js}$$

which yields

$$\frac{d}{dt}\frac{\partial T}{\partial q_s^{\cdot}} = \frac{d}{dt}\left(\frac{\partial T^*}{\partial \pi_j^{\cdot}}\right) a_{js} + A_s, \quad A_s = \frac{\partial T^*}{\partial \pi_j^{\cdot}}\left[\frac{\partial a_{js}}{\partial q_r}(b_{rk}\pi_k^{\cdot} + b_r) + \frac{\partial a_{js}}{\partial t}\right]$$

Using (1.5) we arrive at the following relation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_s^{\cdot}}\right) b_{si} = \frac{d}{dt}\frac{\partial T^*}{\partial \pi_i^{\cdot}} + A_s b_{si}$$

Next, differentiating (1.7) with respect to q_s, we obtain

$$\frac{\partial T}{\partial q_s} = \frac{\partial T^*}{\partial q_s} + B_s, \quad B_s = \frac{\partial T^*}{\partial \pi_j^{\cdot}}\left[\frac{\partial a_{jr}}{\partial q_s}(b_{rk}\pi_k^{\cdot} + b_r) + \frac{\partial a_j}{\partial q_s}\right]$$

and hence

$$\frac{\partial T}{\partial q_s} b_{si} = \frac{\partial T^*}{\partial \pi_i} + B_s b_{si}$$

where the partial derivative in the quasicoordinate π_i denotes the operator $\partial T^*/\partial\pi_i = (\partial T^*/\partial q_s)\, b_{si}$. Substituting the expressions obtained into (1.6), we obtain the required equations of motion

$$\frac{d}{dt}\frac{\partial T^*}{\partial \pi_i^{\cdot}} - \frac{\partial T^*}{\partial \pi_i} + \frac{\partial T^*}{\partial \pi_j^{\cdot}}(\gamma_{ijk}\pi_k^{\cdot} + \gamma_{ij}) = \Pi_i \qquad (i = 1, 2, \ldots, n) \tag{1.8}$$

$$\gamma_{ijk} = b_{si}b_{rk}\left(\frac{\partial a_{js}}{\partial q_r} - \frac{\partial a_{jr}}{\partial q_s}\right)$$

$$\gamma_{ij} = b_{si}\left[b_r\left(\frac{\partial a_{js}}{\partial q_r} - \frac{\partial a_{jr}}{\partial q_s}\right) + \frac{\partial a_{js}}{\partial t} - \frac{\partial a_j}{\partial q_s}\right]$$

The expressions for the coefficients γ_{ijk} and γ_{ij} in these equations are best found using so-called commutation relations /6/, which in the present case have the form

$$d\delta\pi_j - \delta d\pi_j = \gamma_{ijk} d\pi_k \delta\pi_i + \gamma_{ij} dt \delta\pi_i \tag{1.9}$$

and are constructed using expressions (1.3) and (1.4).

Equations (1.8) in quasicoordinates differ from the Boltzmann-Hamel equations in additional terms $\gamma_{ij}(\partial T^*/\partial\pi_j^{\cdot})$, which vanish only when system (1.3) is homogeneous and its coefficient do not depend explicitly on time.

Equations (1.8) describe the dynamics of a system with rolling, with or without slippage. Indeed, if we have rolling with slippage, then there are no equations (1.1) and the quantities $\pi_1^{\cdot}, \pi_2^{\cdot}, \ldots, \pi_n^{\cdot}$, which we shall call the characteristics, can take any values. We determine them as functions of time using system (1.8) which, together with relations (1.3) forms a closed system of differential equations. If on the other hand we have rolling without slippage, then the characteristics $\pi_{m+1}^{\cdot} = \pi_{m+2}^{\cdot} = \ldots = \pi_n^{\cdot} = 0$ and equations of dynamics (1.8) will be formulated, since equations (1.1) will be satisfied only for the first m characteristics $\pi_1^{\cdot}, \pi_2^{\cdot}, \ldots, \pi_m^{\cdot}$.

In the case of a system with rolling represented by a wheeled vehicle, the conditions of rolling without slippage can hold not for all wheels, but for one wheel, two, etc. In this case the last $n-m$ equations of motion (1.8) can be conveniently written in the form

$$\frac{d}{dt}\frac{\partial T^*}{\partial\pi_l^{\cdot}} - \frac{\partial T^*}{\partial\pi_l} + \frac{\partial T^*}{\partial\pi_j^{\cdot}}(\gamma_{ljk}\pi_k^{\cdot} + \gamma_{lj}) = \Pi_l' + R_l \tag{1.10}$$
$$(l = m+1,\ m+2, \ldots, n)$$

after separating from the generalized forces Π_l the forces R_l acting between the wheel and the road.

Further, we shall consider the case when the wheels are rigid and dry frictional forces obeying the Coulomb-Amonton law exist in the areas of contact between the wheel and the supporting surface. In this case the reaction

$$R_l = \begin{cases} -\nu N_l & (\pi_l^{\cdot} > 0) \\ -\nu N_l < R_l < \nu N_l & (\pi_l^{\cdot} = 0) \\ \nu N_l & (\pi_l^{\cdot} < 0) \end{cases} \tag{1.11}$$

provided that the l-th wheel has only a single component $\pi_l^{\cdot}$ of the slippage velocity, ν is the coefficient of sliding friction, and N_l is the normal force acting on the l-th wheel from the direction of the supporting surface.

When the slippage velocity has two components ($\pi_l^{\cdot}$ is the longitudinal and $\pi_{l+1}^{\cdot}$ the transverse component), the reactions R_l and R_{l+1} are given by the relations

$$R_l = -\nu N_l \pi_l^{\cdot}(\pi_l^{\cdot 2} + \pi_{l+1}^{\cdot 2})^{-1/2}, \quad R_{l+1} = -\nu N_l \pi_{l+1}^{\cdot}(\pi_l^{\cdot 2} + \pi_{l+1}^{\cdot 2})^{-1/2} \tag{1.12}$$

provided that $\pi_l^{\cdot} \neq 0$ and (or) $\pi_{l+1}^{\cdot} \neq 0$, and take any value within the region $(R_l^2 + R_{l+1}^2)^{1/2} < \nu N_l\ (N_l > 0)$, provided that $\pi_l^{\cdot} = \pi_{l+1}^{\cdot} = 0$.

Thus when the l-th wheel rolls without slippage, the quantity R_l, or respectively R_l and R_{l+1} in (1.10) are unknown functions of time, like the phase variables, and are obtained by integrating the differential equations of motion.

The system of equations (1.10) can be solved for the accelerations $\pi_l^{\cdot\cdot}$. After eliminating the remaining accelerations $\pi_j^{\cdot\cdot}$, in these equations, we can use the equations of dynamics (1.8) written for the first m quasicoordinates, to write it in the standard Cauchy form

$$\pi_l^{\cdot\cdot} = f_l(q_1, \ldots, q_n, \pi_1^{\cdot}, \ldots, \pi_n^{\cdot}, R_{m+1}, \ldots, R_n) \tag{1.13}$$
$$(l = m+1,\ m+2,\ \ldots,\ n)$$

The equations obtained are used in Sect.2 to derive the conditions for transferring from the rolling of the wheel without slippage to rolling with slippage, and the conditions of the reverse process. Since $\pi_l^{\cdot}$ is a component of the slippage velocity, every equation of system (1.13) expresses the dependence of the rate of change of the l-th component on the values of the phase variables, since according to (1.11) and (1.12) the quantities $R_l\ (l = m+1, m+2, \ldots, n)$ are also functions of the phase coordinates irrespective of the mode of rolling.

2. Structure of the phase space of the system with rolling. The conditions for transfer from one mode of motion to the other. The motion of a system with rolling becomes geometrically clear when the phase space is brought into consideration and the special features of its structure are explained. From Sect.1 it follows that in the case of motion with slippage the state of the system at any instant t is determined by $2n$ parameters: n generalized coordinates $q_1, q_2, \ldots, q_n$ and n characteristics $\pi_1^\cdot, \pi_2^\cdot, \ldots, \pi_n^\cdot$. Therefore, equations (1.3) and (1.8) describe the motion of a representative point in extended $2n+1$ -dimensional phase space $\Phi(q, \pi^\cdot, t)$.

Equations (1.1) describing the rolling of the system in question without slippage, can be regarded as equations of some $n+m+1$ -dimensional hyperplane Π

$$\pi_{m+1}^\cdot = 0,\ \pi_{m+2}^\cdot = 0,\ \ldots,\ \pi_n^\cdot = 0 \tag{2.1}$$

in the phase space Φ. This implies that during rolling without slippage the representative point moves in the phase space Φ along the hyperplane (2.1), obeying the equations of motion (1.8) written for the first characteristics $\pi_1^\cdot, \pi_2^\cdot, \ldots, \pi_m^\cdot$, only, and relations (1.3). Relations (2.1) must be taken into account in all these expressions.

If a system with rolling is represented by a wheeled vehicle, the hyperplane consists of the intersection of the branches $\Pi^1, \Pi^2, \ldots, \Pi^r$, whose number r is determined by the number and the distribution of the wheels. The motion of the representative point along one of the branches $\Pi^1, \Pi^2, \ldots, \Pi^r$ corresponds to rolling without slippage of one (or several) wheel(s), and the motion along the hyperplane Π the rolling without slippage of all wheels simultaneously.

Let us consider the special features of the structure of the decomposition of the phase space Φ into trajectories related to the presence of the branch Π^s $(s = 1, 2, \ldots, r)$, in the case when the wheels are rigid, and dry friction forces obeying the Coulomb-Amonton law appear during their contact with the supporting surface. Since the dependence of the dry friction force on the velocity of slippage v can be represented by a curve with a first-order discontinuity at $v = 0$, the differential equations describing the motion of the representive point in different regions of the phase space Φ, produced by partitioning the latter with the hyperplanes $\Pi^1, \Pi^2, \ldots, \Pi^r$, will themselves be different. The equations may contain the quantities N_s i.e. the normal pressure forces between the interacting bodies, and the forces may be constant (i.e. functions of the physical parameters of the system), or may also depend on the phase variables.

Normally, the quantities N_s are finite and positive $(N > 0$ if the interacting bodies exert pressure on each other). When N_s vanishes, it simple means that the corresponding bodies are no longer in contact. We shall assume that when N_s have finite values, the usual conditions ensuring the existence and uniqueness of the solutions of differential equations with given initial values of the variables, are satisfied in all regions of the phase space Φ of the system with rolling discussed here.

It may happen however that when the parameters and the phase variables are in a certain ratio, the normal pressure N_s becomes negative, passing from positive to negative values not through zero, but through infinity. (Such a situation arises when the coefficient of N_s vanishes, or when the determinant of the system of equations used to determine the normal pressure force vanishes). In this special case (the Penleve paradox /7/) the initial hypotheses of the classical mechanics of solids are found to be insufficient to determine the motion of the system with rolling in question. The physical reason for the Penleve paradox is connected with a wedging effect, which leads, within the framework of the model of a perfectly rigid body, to the reaction forces increasing to infinity. The passage of the force N_s from positive to negative values through infinity is expressed by the fact that a manifold of points exists in phase space, in which the phase velocity field undergoes second-order discontinuities. Such a form of violation of the regularity of the vector velocity field in phase space can serve as an indication that we have the Penleve paradox.

Subsequent discussions will deal with systems with rolling in which the Penleve paradox does not occur.

Let us assume, to be specific that the branch Π^s corresponds to rolling of the s-th wheel without slippage. If by virtue of the constraints imposed on the system the s-th wheel is forced to roll in such a manner that there is no side or longitudinal slippage, then the vector $\mathbf{v}_s$ of the wheel slippage velocity will have a single component which we shall denote by $\pi_s^\cdot$. The hyperplane $\Pi^s(\pi_s^\cdot = 0)$ will divide the phase space Φ into two regions

$$\Phi_+(\pi_s^\cdot > 0) \quad \text{and} \quad \Phi_-(\pi_s^\cdot < 0)$$

According to the Coulomb-Amonton law the sliding friction froce takes, in the region Φ_+, a value depending on the normal pressure, and in the region Φ_- the same value but with opposite sign. Therefore, Eqs.(1.8) describing the motion of the representative point in Φ_+ and Φ_-, will be different in these regions. The solutions of Eqs.(1.8) must match, when passing from one region to the other, by virtue of the continuity of the phase coordinates.

The motion of the representative point along the hyperplane Π^s is determined by Eqs. (1.8) written for the quasicoordinates $\pi_1, \ldots, \pi_{s-1}, \pi_{s+1}, \ldots, \pi_n$, in which the relation $\pi_s^\cdot = 0$ has been taken into account. The motion, however, is realized only in the region G^s of the hyperplane Π^s, stable with respect to the deviations from Π^s. The necessary and sufficient condition of stability of the region G^s is that the distribution of the phase trajectories in a small neighbourhood of the hyperplane Π^s, must be such, that the representative point moving along these trajectories arrrives at the hyperplane from the region Φ_+, as well as from Φ_-. We shall call such a distribution the link-up of the phase trajectories. Thus the connected region of the link-up of the phase trajectories defines the region G^s on the hyperplane Π^s.

The boundary Γ^s of the region G^s will consist of Γ_+^s and Γ_-^s. If the representative point moving into the region G^s reaches the boundary Γ_+^s, it passes to the region Φ_-, and after reaching the boundary Γ_-^s it goes into the region Φ_+. All this implies that the determination of the boundaries of G^s solves completely the problem of determining the conditions under which the rolling of a wheel without slippage changes to rolling with slippage, and the conditions of the reverse process.

The mathematical determination of the region G^s now reduces to simultaneous satisfaction of the following two inequalities:

$$\lim_{\pi_s^{\cdot}\to 0} \pi_s^{\cdot\cdot} \leqslant 0, \qquad \lim_{\pi_s^{\cdot}\to -0} \pi_s^{\cdot\cdot} \geqslant 0 \tag{2.2}$$

Let $\pi_s^{\cdot\cdot} = f_s(\pi_1^{\cdot}, \ldots, \dot{\pi}_{s-1}, \pi_s^{\cdot}, \dot{\pi}_{s+1}, \ldots, \pi_n^{\cdot}, N_s)$ be the equation of system (1.13) constructed for the quasicoordinate π_s, where N_s is the normal load on the s-wheel. Then inequalities (2.2) will take the form

$$\begin{aligned} &f_s(\pi_1^{\cdot}, \ldots, \dot{\pi}_{s-1}, 0, \dot{\pi}_{s+1}, \ldots, \pi_n^{\cdot}, N_s) \leqslant 0 \\ &f_s(\pi_1^{\cdot}, \ldots, \dot{\pi}_{s-1}, 0, \dot{\pi}_{s+1}, \ldots, \pi_n^{\cdot}, -N_s) \geqslant 0 \end{aligned} \tag{2.3}$$

Converting (2.3) into an equality we obtain the equations of the boundaries Γ_+^s and Γ_-^s of the region G^s.

In the case in question the system with rolling is related to the well-known class of systems with discontinuous right-hand sides /8—10/. The results obtained here apply also to systems with rolling. We must only remember that unlike relay systems of automatic control, the equations of motion of the systems with rolling are usually non-linear in the regions Φ_+ and Φ_-.

In the case of the rolling of a rigid wheel, the vector v_s of the rate of slippage has usually two components, the transverse $\pi_s^{\cdot}$ and the longitudinal $\pi_{s+1}^{\cdot}$. We shall consider a three-dimensional cross-section Φ_3 of the phase space Φ, plotting the quantity $\pi_s^{\cdot}$, along the abscissa and $\pi_{s+1}^{\cdot}$, along the ordinate, and using as the z coordinate e.g. the angular velocity ω_s of the natural rotation of the wheel. The straight line $\pi_s^{\cdot} = 0$, $\pi_{s+1}^{\cdot} = 0$, i.e. the axis ω_s, will represent the hyperplane Π^s in Φ_3.

Introducing a cylindrical $v_s, \vartheta_s, \omega_s$ coordinate system by means of the relations $\pi_s^{\cdot} = v_s \cos\vartheta_s$, $\pi_{s+1}^{\cdot} = v_s \sin\vartheta_s$ and writing the equations of motion in the new phase variables $v_s, \vartheta_s, \omega_s$, we obtain the equation $v_s^{\cdot} = F_s(v_s, \vartheta_s, \omega_s, \ldots)$ for v_s. From the condition that $v_s^{\cdot} \leqslant 0$ as $v_s \to 0$, it follows that the region G^s is defined on the ω_s axis by the inequality

$$F_s(0, \vartheta_s, \omega_s, \ldots) \leqslant 0 \tag{2.4}$$

which must hold in the interval $0 \leqslant \vartheta_s < 2\pi$ for all values of ϑ_s. Since the left-hand side of inequality (2.4) also contains all remaining phase variables which were assumed fixed when dicussing the three-dimensional cross-section, changing (2.4) into an equality yields the equation of the boundary Γ^s of G^s on the hyperplane Π^s in the phase space Φ. In this case the motion of the representative point in the region G^s on the hyperplane Π^s is described by the equation (1.8) written for the quasicoordinates $\pi_1, \ldots, \pi_{s-1}, \pi_{s+2}, \ldots, \pi_n$ taking the relations $\pi_s^{\cdot} = 0$, $\pi_{s+1}^{\cdot} = 0$ into account.

The present case differs from the two-relay system thus. In this case of the relay system the region G has a meaning on each hyperplane: $\pi_s^{\cdot} = 0$ and $\pi_{s+1}^{\cdot} = 0$. The motion of the representative point on one of these regions maps a so-called sliding mode of one of the relays, and the motion along their intersection maps a simultaneous sliding mode of both relays. In the case of a wheel, only the region G at the intersection of these hyperplanes, where the rolling of the wheel without slippage is mapped, has any meaning.

Notes. 1°. The sliding modes in discontinuous systems have been studied by many workers /9, 10/. An interpretation of a sliding mode in phase space has already been given in /11/. The present paper, however, appears to be the first to employ an analogous approach to the study of systems with rolling, taking slippage into account.

2°. According to (1.13) the left-hand sides of inequalities (2.3) and (2.4) contain the reactions R_l ($l = m+1, m+2, \ldots, n$), i.e. they depend on the mode of rolling of elastic wheels, e.g. of the l-th wheel. Therefore, we must distinguish here between two cases: 1) the l-th

wheel rolls with slippage, and 2) the l-th wheel rolls without slippage.

In the first case we must substitute into inequalities (2.3) and (2.4) the expressions (1.11) if the slippage velocity of the l-th wheel has only a single component, and the expression (1.12) when it has two components. In the second case the quantity R_l and the corresponding R_l and R_{l+1} must be expressed in terms of the phase variables using the following procedure. When the l-th wheel has a single component of the slippage velocity, we must put $\pi_l^{\cdot} = \pi_{l+1}^{\cdot\cdot} = 0$ in Eq.(1.10) written for the quasicoordinate π_l, and eliminate the remaining accelerations $\pi_j^{\cdot\cdot}$, appearing in this equation by means of the other equations of dynamics (1.8). When the slippage velocity of the l-th wheel has two components, we must write $\pi_l^{\cdot} = \pi_l^{\cdot\cdot} = \pi_{l+1}^{\cdot} = \pi_{l+1}^{\cdot\cdot} = 0$ in equations (1.10) written for the quasicoordinate π_l and π_{l+1}, and eliminate the remaining accelerations $\pi_j^{\cdot\cdot}$, in these equations using the other equations of dynamics (1.8). As a result, the quantities R_l and R_{l+1} turn out to be the functions of the phase variables and should be substituted into inequalities (2.3) and (2.4) in the case when the l-the wheel rolls without slippage.

3°. The proposed theory can be applied to the case of rolling of a wheel with a deformable tyre, even within the framework of the phenomenological theory (such as the Keldysh theory), only under the specified conditions. Since we have here, instead of a point contact, an area of contact between the wheel and the supporting surface, we can have partial slippage of the tyre, or a moment of sliding frictional force may appear as a result of rotation of the contact area. In cases when these additional factors can be neglected and the slippage velocity field deviates little from the uniform, the theory developed here can also be applied to a wheel with a deformable tyre.

Example 1. Let a perfectly rigid wheel, whose plane is vertical, roll along a horizontal rough line (Fig.1). A constant angular momentum M is applied to the wheel and a viscous frictional force with coefficient h acts on the wheel. Then the wheel motion is described by the equations

$$mx^{\cdot\cdot} + hx^{\cdot} = R, \quad mk^2\varphi^{\cdot\cdot} = M - rR$$

where m is the mass and r is the radius of the wheel, k is the central radius of gyration, x is the coordinate of the wheel's centre, φ is the angle of rotation and R is the horizontal component of the force of interaction between the wheel and the supporting straight line. If the wheel rolls without slippage, then the following relation representing the equation of kinematic constraint connects the variables used:

$$x^{\cdot} - r\varphi^{\cdot} = 0$$

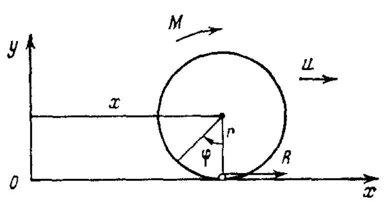

Fig.1

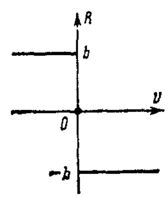

Fig.2

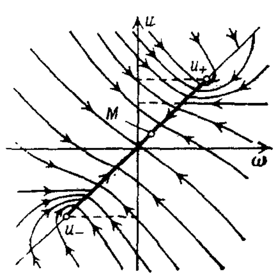

Fig.3

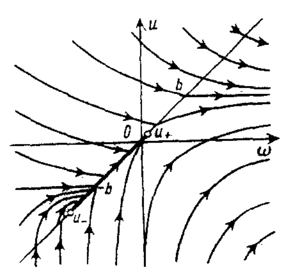

Fig.4

Let us introduce the dimensionless coordinates

$$x_0 = xr^{-1}, \ t_0 = m^{-1}ht, \ R_0 = mr^{-1}h^{-2}R, \ M_0 = mr^{-2}h^{-2}M$$
$$a = k^2r^{-2}$$

and write $u = x_0^{\cdot}$, $\omega = \varphi^{\cdot}$. The equations of dynamics in the new variables take the form

$$u^{\cdot} + u = R_0, \quad a\omega^{\cdot} = M_0 - R_0 \tag{2.5}$$

and the equations of kinematic constraints become

$$v \equiv u - \omega = 0 \tag{2.6}$$

Let R be the force of dry friction. The graph in Fig.2 shows, according to the Coulomb-Amonton law, the dependence of R_0 on the slippage velocity v with b denoting the magnitude of the sliding friction. From (2.5) and (2.6) and the graph of the function $R_0 = R_0(v)$, it follows that the rolling of the wheel is described by the motion of the representative point in the phase plane $\Phi(\omega, u)$, which splits into two regions, Φ_+, in which $v > 0$, and Φ_-, in which $v < 0$. the straight line Π on which $v = 0$, forms the boundary between them. The motion of the representative point in the phase plane Φ is described by the equations

$$\begin{aligned} &u^{\cdot} + u = -b, \quad a\omega^{\cdot} = M_0 + b \quad (\Phi_+) \\ &(1 + a)\,\omega^{\cdot} + \omega = M_0 \qquad (\Pi) \\ &u^{\cdot} + u = b, \quad a\omega^{\cdot} = M_0 - b \quad (\Phi_-) \end{aligned}$$

Fig.3 shows the decomposition of the plane Φ into trajectories in the case $0 < M_0 < b$ where the values $u_{\pm} = \pm[(1 + a)\,b \mp M_0]a^{-1}$ are obtained from the condition that the phase trajectories are in contact with the bisectrix $u = \omega$ in the region Φ_+ and Φ_-. The rolling of the wheel without slippage is represented by the motion of the phase point in the region Π on the segment $u_- \leqslant u \leqslant u_+$, where we have a stable singularity $u = \omega = M_0$. The phase pattern shown in Fig.3 shows that in the case in question and for any intial conditions, the wheel will eventually end in a stable stationary motion, i.e. rolling without slippage.

In the case when $M_0 > b$ we have a phase pattern shown in Fig.4. Here, for any initial conditions the representative point in the phase plane $\Phi(\omega, u)$ will approach asymptotically, with time the motion along the straight line $u = b$ with velocity $(M_0 - b)\,a^{-1} > 0$. At the same time the quantity ω will increase to infinity. Such a motion of the representative point corresponds to a wheel rolling with slippage, so that the velocity of the wheel centre tends to a finite value $u = b$, and the angular velocity of rotation increases to infinity.

Example 2. We shall consider the problem of the motion of a motorcycle taking sideways slippage of the wheels into account, under the following simplifying assumptions: the mass of the rigid wheels is negligible compared with the mass of the rider and chassis. We shall regard the rider and the chassis as a single rigid body of mass m, with principal central moments of inertia A and B. The velocity V of the longitudinal motion of the motorcycle and the angle of rotation of the steering ψ are given functions of time; the quantity ψ, and the angle of inclination of the chassis χ, the rate of transverse displacement of the centre of mass u and the projections ω_1, ω_2 of the instantaneous angular velocity of the body on the principal direction of the central energy ellipsoid are all fairly small. Then, with both wheels sliding sideways, the equations of motion linearized with respect to small quantities, have the form (see Fig.5)

$$\begin{aligned} &mu^{\cdot} = -mV(\omega_1 \sin\alpha + \omega_2 \cos\alpha) + F_1 + F_2 \\ &A\omega_1^{\cdot} = H\cos\alpha + M_1F_1 + K_1F_2 \\ &B\omega_2^{\cdot} = -H\sin\alpha - M_2F_1 - K_2F_2 \\ &\chi^{\cdot} = \omega_1\cos\alpha - \omega_2\sin\alpha; \quad H = mg\,(h\chi - c_1lc^{-1}\psi) \end{aligned} \tag{2.7}$$

Here F_1, F_2 are the transverse reactions of the road on the rear and front wheel at the points of contact between the wheels and the road. Assuming that the reactions are dry friction forces, we have

$$F_{1,2} = -\nu N_{1,2}\,\mathrm{sign}\,u_{1,2}, \quad \text{if } u_{1,2} \neq 0 \tag{2.8}$$

and any value in the interval $(-\nu N_{1,2} < F_{1,2} < \nu N_{1,2})$, provided that $u_{1,2} = 0$; ν is the coefficient sliding friction $N_1 = mglc^{-1}$, $N_2 = mg\,(c - l)\,c^{-1}$ are the normal pressure forces, and u_1, u_2 are the transverse sliding velocities of the rear and front wheel respectively, described by the expressions

$$\begin{aligned} &u_1 = u + M_1\omega_1 - M_2\omega_2 \\ &u_2 = u + K_1\omega_1 - K_2\omega_2 - \Psi, \\ &\Psi = c_1\psi^{\cdot} - V\psi\cos\lambda \\ &M_1 = h\cos\alpha - l\sin\alpha, \\ &K_1 = h\cos\alpha + (c - l)\sin\alpha \\ &M_2 = h\sin\alpha + l\cos\alpha, \\ &K_2 = h\sin\alpha - (c - l)\cos\alpha \end{aligned} \tag{2.9}$$

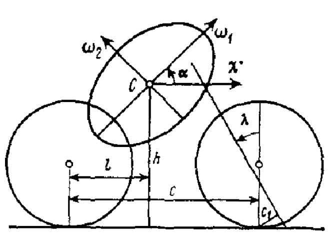

Fig.5

Equations (2.1) describe the motion of the representative point in four-dimensional phase space $\Phi(u, \omega_1, \omega_2, \chi)$.

Let us consider possible special cases of the motion of a motorcycle.

Only the rear wheel slips. In this case the representative point moves in the phase space Φ on the hyperplane $u_2 = 0$ according to the equations of motion (2.7), from which we must eliminate F_2 and use the relation

$$u_2 \equiv u + K_1\omega_1 - K_2\omega_2 - \Psi = 0$$

The motion on the hyperplane $u_2 = 0$ occurs in the region G^2 defined by the inequalities

$$\lim_{u_2\to+0} u_2^{\cdot} \leqslant 0, \qquad \lim_{u_2\to-0} u_2^{\cdot} \geqslant 0$$

According to (2.9) and (2.7) we have an expression for $u_2^{\cdot}$ and obtain the first inequality

$$\begin{gathered} H\,(A^{-1}K_1 \cos\alpha + B^{-1}K_2 \sin\alpha) + F_1\,(m^{-1} + A^{-1}M_1K_2 + \\ B^{-1}M_2K_2) - \nu N_2\,(m^{-1} + A^{-1}K_1^2 + B^{-1}K_2^2) - \Psi^{\cdot} - \\ V\,(\omega_1 \sin\alpha + \omega_2 \cos\alpha) \leqslant 0 \end{gathered} \tag{2.10}$$

where F_1 is given by (2.8). The second inequality separating out the region G^2 is obtained from (2.10) by replacing N_2 by $-N_2$ and the sign $\leqslant$ by $\geqslant$.

Only the front wheel slips. In this case the representative point moves along the hyperplane $u_1 = 0$ in the region G^1 defined by the inequalities

$$\begin{gathered} H\,(A^{-1}M_1 \cos\alpha + B^{-1}M_2 \sin\alpha) + F_2\,(m^{-1} + A^{-1}M_1K_1 + \\ B^{-1}M_2K_2) - \nu N_1\,(m^{-1} + A^{-1}M_1^2 + B^{-1}M_2^2) - \\ V\,(\omega_1 \sin\alpha + \omega_2 \cos\alpha) \leqslant 0 \end{gathered} \tag{2.11}$$

where F_2 is given by (2.8). The second inequality is obtained from (2.11) by replacing N_1 by $-N_1$ and the sign $\leqslant$ by $\geqslant$.

Both wheels roll without slipping. In this case the representative point moves in the region representing the intersection of two hyperplanes, $u_1 = 0$ and $u_2 = 0$. Using these equations and eliminating F_1, F_2 from (2.7), we arrive at the equation of motion of a motorcycle without slipping

$$\begin{gathered} cJ_1\chi^{\cdot\cdot} - mg\,ch\,\chi - c_1J_{12}\psi^{\cdot\cdot} - (J_{12}\cos\lambda + mc_1h)\,V\psi^{\cdot} + \\ (mgc_1l - mhV^2\cos\lambda - J_{12}V^{\cdot}\cos\lambda)\,\psi = 0 \\ J_1 = A\cos^2\alpha + B\sin^2\alpha + mh^2 \\ J_{12} = (B - A)\sin\alpha\cos\alpha + mhl \end{gathered} \tag{2.12}$$

When $V = \mathrm{const}$, the above equations becomes identical with the first equation of (1.5) in /12/, provided that we neglect in (1.5) the mass of the front part of the bicycle and the moments of inertia of the wheels.

Motion in the region $u_1 = 0, u_2 = 0$ occurs when the four inequalities (2.10) and (2.11) are satisfied simultaneously, in which

$$\begin{gathered} F_1 = c^{-2}h^{-1}J_1^{-1}\,\{[J_{12}\,(BK_1\sin\alpha - AK_2\cos\alpha) - \\ J_1\,(AK_2\sin\alpha + BK_1\cos\alpha)]\Psi^{\cdot} - mhV\,(BK_1\sin\alpha - \\ AK_2\cos\alpha)\,\Psi - cH\,[J_1\,(c - l) + AK_2\cos\alpha - BK_1\sin\alpha]\} \\ F_2 = c^{-2}h^{-1}J_1^{-1}\,\{[J_{12}\,(AM_2\cos\alpha - BM_1\sin\alpha) + \\ J_1\,(AM_2\sin\alpha + BM_1\cos\alpha)]\,\Psi^{\cdot} + mhV\,(AM_2\cos\alpha - \\ BM_1\sin\alpha)\,\Psi + cH\,(AM_2\cos\alpha - BM_1\sin\alpha - lJ_1)\} \end{gathered}$$

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MOTION OF A HEAVY RIGID BODY ON A HORIZONTAL PLANE WITH VISCOUS FRICTION*

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The motion of an arbitrary, heavy rigid body on a horizontal plane with viscous friction is considered. It is shown that the limit set of trajectories of motion is represented by the set of motions of this body on a perfectly smooth surface without slippage. The set represents the intersection of the manifolds of steady motions of the body on perfectly smooth and perfectly rough surfaces and, depending on the dynamic and geometrical characteristics of the body, it may include the states of equilibrium, steady rotations about the vertical, uniform rolling motions along a fixed straight line, and regular processions. Examples of the motion of specific bodies are discussed.

1. Let a rigid body move along a fixed horizontal plane, touching it at a single point P of its surface. The motion takes place in a uniform gravitational field. The supporting plane is defined in the fixed $O\xi\eta\zeta$ coordinate system by the equation $\zeta = 0$, and the $O\zeta$ axis points vertically upwards. We shall introduce a right $Gxyz$, coordinate system rigidly fixed to the body, direct its axes along the principal central moments of inertia of the body, and place its origin at the centre of gravity of the body. We shall define the position of the body by the coordinates ξ, η, ζ of its centre of gravity in the fixed coordinate system, and the Euler angles ψ, θ, φ, defining the orientation of the body in absolute space. The coordinate ζ will be a known function of the angles θ and φ, i.e. $\zeta = f(\theta, \varphi) > 0$. We shall assume that the function f is a fairly smooth function of its arguments and such that the body can touch the supporting plane only at a single point of its surface. We will denote the projection of the centre of gravity G onto the supporting plane by Q. Henceforth, A, B, C will denote the moments of inertia of the body about the axes Gx, Gy, Gz, m is the mass of the body and g is the acceleration due to gravity.

We have the following expression for $\zeta^{\cdot}$:

$$\zeta^{\cdot} = \rho_\theta \theta^{\cdot} + \rho_\varphi \varphi^{\cdot}, \quad \rho_\theta = \partial f / \partial\theta, \quad \rho_\varphi = \partial f / \partial\varphi \tag{1.1}$$

The critical points of the function $f(\theta, \varphi)$ correspond to the positions of equilibrium of the body in the plane $(P = Q, \rho_\theta = \rho_\varphi = 0)$. Any body has at least two different positions of equilibrium. This follows from the fact that a function on a sphere has at least two critical points.

Let us assume that the body is acted upon at the point P from the direction of the plane by the viscous force $\mathbf{F} = -mk\mathbf{V}_P$, where $\mathbf{V}_P$ is the velocity of the point P of the body in the fixed coordinate system, and $k > 0$ is the coefficient of friction. Then the following expression can be obtained for the total energy $E > 0$ of the body:

$$dE / dt = -mkV_P^2 \tag{1.2}$$

From (1.2) we see that E does not increase and V_P tends to zero with time, i.e. the body has a tendency to avoid slipping /1/. Therefore we have

$$\lim_{t\to\infty} E(t) = E^* \geqslant \nu > 0, \quad \nu = mg \min_{\theta,\varphi} f(\theta, \varphi) \tag{1.3}$$

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